

Generalized Liar's Dominating Set in Graphs

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Abstract

In this article, we study generalized liar's dominating set problem in graphs. Let $G = (V, E)$ be an undirected simple graph. The generalized liar's dominating set, called as the distance- d (m, ℓ) -liar's dominating set, is a subset $L \subseteq V$ such that (i) each vertex in V is distance- d dominated by at least m vertices in L , and (ii) each pair of distinct vertices in V is distance- d dominated by at least ℓ vertices in L , where m, ℓ, d are positive integers and $m < \ell$. Here, a vertex v is distance- d dominated by another vertex u means the shortest path distance between u and v is at most d in G .

We first consider distance-1 (m, ℓ) -liar's dominating set problem and prove that it is NP-complete. Next, we consider distance- d (m, ℓ) -liar's dominating set problem and show that it is also NP-complete. These liar's dominating set problems are generalized version of liar's dominating set problem on [11] studied only distance-1 $(2, 3)$ -liar's dominating set problem till now. We also prove that (i) distance-1 (m, ℓ) -liar's dominating set problem cannot be approximated within a factor of $(\frac{1}{2} - \varepsilon) \ln |V|$ for any $\varepsilon > 0$, unless $NP \subseteq DTIME(|V|^{O(\log \log |V|)})$, and (ii) distance- d (m, ℓ) -liar's dominating set problem cannot be approximated within a factor of $(\frac{1}{4} - \varepsilon) \ln |V|$ for any $\varepsilon > 0$, unless $NP \subseteq DTIME(|V|^{O(\log \log |V|)})$.

1 Introduction

Let $G = (V, E)$ be a simple connected and undirected graph. For a vertex $v \in V$, the closed neighborhood of v in G is denoted by $N_G^1[v]$ and is defined as $N_G^1[v] = \{u \in V \mid (u, v) \in E\} \cup \{v\}$. A dominating set of G is a subset D of V such that every vertex in V is in either D or adjacent to at least one vertex in D . In other words, $|N_G^1[v] \cap D| \geq 1$ for each $v \in V$. The vertices in D are called as dominators and the rest are called as dominatees. A dominator dominates all its neighbors and itself. The dominating set problem asks to find a dominating set of minimum cardinality. A k -tuple dominating set of G is a dominating set with the restriction that every vertex in V must be dominated by at least k vertices in the dominating set, i.e., $|N_G^1[v] \cap D| \geq k$ for each $v \in V$. The goal of the

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k -tuple dominating set problem is to find a k -tuple dominating set of minimum size. A liar's dominating set D of G is a dominating set having the following two properties: (i) D is a 2-tuple dominating set, and (ii) $|(N_G^1[u] \cup N_G^1[v]) \cap D| \geq 3$, for every pair of distinct vertices u and v in V . The objective of the liar's dominating set problem is to find a liar's dominating set of minimum size in a given graph G .

Given a simple connected undirected graph $G = (V, E)$, $\delta_G(v_i, v_j)$ denotes the length of a shortest path between the vertices v_i and v_j in G . For an integer $d > 0$, the distance- d neighborhood of a vertex $v_i \in V$ is denoted by $N_G^d[v_i]$ and defined as $N_G^d[v_i] = \{v_j \in V \mid \delta_G(v_i, v_j) \leq d\}$. A distance- d (m, ℓ) -liar's dominating set (distance- d (m, ℓ) -LDS) of G is a subset $L \subseteq V$ such that (i) for every $v_i \in V$, $|N_G^d[v_i] \cap L| \geq m$, and (ii) for every pair of distinct vertices $v_i, v_j \in V$, $|(N_G^d[v_i] \cup N_G^d[v_j]) \cap L| \geq \ell$, where m, ℓ, d are positive integers and $m < \ell$. The objective of the distance- d (m, ℓ) -LDS problem is to find a minimum size distance- d (m, ℓ) -LDS in a given graph G , and we call this problem as the minimum distance- d (m, ℓ) -liar's dominating set problem. In Fig. 1, the set of vertices $\{e, f, i\}$ form a distance-3 $(2, 3)$ -LDS. Note that liar's dominating set is a distance- d (m, ℓ) -liar's dominating set with $d = 1, m = 2$, and $\ell = 3$.

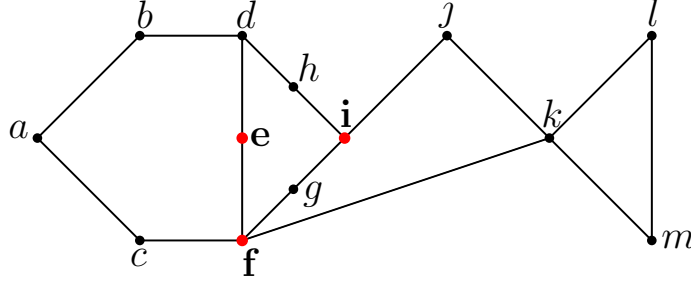


Figure 1: The set $\{e, f, i\}$ is the distance-3 $(2, 3)$ -liar's dominating set.

Our interest in the problem arises from the following real-time application. Assume each vertex in a graph $G = (V, E)$ is a possible location for an intruder such as a thief, a saboteur, a fire or some possible fault. Assume also that there is exactly $\min(\ell - m, \lceil \ell/2 \rceil - 1)$ intruders in the system represented by G . A protection device placed at a vertex v is assumed to be able to (1) detect the intruder at any vertex in its closed distance- d neighborhood $N_G^d[v]$, and (2) report the vertex $u \in N_G^d[v]$ at which the intruder is located. We are interested in deploying protection devices at a minimum number of vertices so that the intruder can be detected and identified correctly. This can be solved by finding a minimum cardinality m -tuple dominating set, say D , of G and deploying protection devices at all the vertices of D . If any one protection device can fail to detect the intruder, then to correctly detect and identify the intruder one needs to place the protection devices at all the vertices of a minimum cardinality $2m$ -tuple dominating set of G . Now it may so happen that all the protection devices detect the intruder location correctly but while reporting some of these protection devices can misreport or lie (either deliberately or through a transmission error) about the intruder location. Assume that at most $\min(\ell - m, \lceil \ell/2 \rceil - 1)$

protection devices in the closed distance- d neighborhood of an intruder location can lie. Under these circumstances, to protect the network we have to install the protection devices at all the vertices of a minimum distance- d (m, ℓ) -liar's dominating set.

2 Related work

In 2009, Slater [11] first introduced minimum distance-1 $(2, 3)$ -LDS problem known as the minimum liar's dominating set (LDS) problem in the literature. He proved that the problem is NP-complete for general graphs and obtained a lower bound for the liar's domination number on trees by proving any minimum distance-1 $(2, 3)$ -LDS cardinality of a tree having n (≥ 3) vertices lies between $\frac{3}{4}(n+1)$ and n . In the same paper, Slater observed for a subclass of trees for which there exist only one distance-1 $(2, 3)$ -LDS which is the entire vertex set. He also proved that for a graph $G = (V, E)$ having n vertices and m edges $\gamma_{LR}(G) \geq \frac{3}{4}(2n - m)$, where $\gamma_{LR}(G)$ is the cardinality of a liar's dominating set of minimum size in G , and $\gamma_{LR}(G) \geq (6/(3\Delta + 2))n$, where Δ is maximum degree of a vertex in G . For a tree T having n vertices, $\gamma_{LR}(T) = n$ if and only if each $v \in V(T)$ is an endpoint or at least one component of $T - v$ has cardinality at most two. Later, Roden and Slater [10] characterized distance-1 $(2, 3)$ -LDS cardinality on tree classes which is equal to $\frac{3}{4}(n+1)$. They showed that even for bipartite graphs the minimum distance-1 $(2, 3)$ -LDS problem is NP-hard. They have even proved some results for complete bipartite graph $K_{a,b}$ with $1 \leq a \leq b$ as follows: (i) $\gamma_{LR}(K_{1,n-1}) = n$, (ii) $\gamma_{LR}(K_{2,b-1}) = b+1$, and (iii) $\gamma_{LR}(K_{a,b}) = \min\{a+1, 6\}$ for $3 \leq a \leq b$.

For different graph classes like split graphs and chordal graphs Panda and Paul [6] proved its NP-hardness and also proposed a linear time algorithm to compute a minimum distance-1 $(2, 3)$ -LDS in trees.

Panda et al. [9] studied the approximability of the problem in general graphs and given an $O(\ln \Delta)$ -factor approximation algorithm, where Δ is the maximum degree of the given graph. For proper interval graphs also Panda and Paul [7] considered the problem and proposed a linear time algorithm. They also studied the minimum distance-1 $(2, 3)$ -LDS problem for bounded degree graphs, and p -claw free graphs [9]. Sterling [12] presented bounds on liar's domination number by considering the problem on two-dimensional grid graphs.

Alimadadi et al. [1] provided the characterization of graphs and trees for which the distance-1 $(2, 3)$ -LDS cardinality is $|V|$ and $|V| - 1$, respectively. The authors observed that a connected graph G with number of vertices $n \geq 3$ has distance-1 $(2, 3)$ -LDS cardinality n if and only if every vertex v in G satisfies at least one of the following conditions (i) $\deg(v) = 1$, (ii) at least one component of $G \setminus \{v\}$ has at most two vertices, (iii) v belongs to an end-block (a block having at most one cut vertex of G) having 3 vertices. For connected graphs with girth (the length of a shortest cycle) at least five, they obtained an upper bound for the ratio between the distance-1 $(2, 3)$ -LDS cardinality and the 2-tuple dominating set cardinality. Panda and Paul [5, 8] studied variants of distance-1 $(2, 3)$ -

LDS, namely, connected distance-1 $(2, 3)$ -LDS and total distance-1 $(2, 3)$ -LDS. A *connected distance-1 $(2, 3)$ -LDS* is a distance-1 $(2, 3)$ -LDS whose induced subgraph is connected. A *total distance-1 $(2, 3)$ -LDS* is a dominating set L with the following two properties (i) for every $v \in V$, $|(N_G^1[v] \setminus \{v\}) \cap L| \geq 2$, and (ii) for every distinct pair of vertices u and v , $|(N_G^1[u] \setminus \{u\}) \cup (N_G^1[v] \setminus \{v\}) \cap L| \geq 3$. The objective of both problems is to find connected distance-1 $(2, 3)$ -LDS and total distance-1 $(2, 3)$ -LDS of minimum size, respectively. They proved that both problems are NP-hard and proposed $O(\ln \Delta)$ -factor approximation algorithms. They also proved that the problems are APX-complete for graphs with maximum degree 4. Jallu and Das [3] studied the geometric version of the minimum distance-1 $(2, 3)$ -LDS problem, namely, Euclidean liar's dominating set problem and presented constant factor approximation algorithms. Recently, Jallu et al. [4] proved that the minimum distance-1 $(2, 3)$ -LDS problem is NP-hard on unit disk graphs and presented an approximation scheme for the same.

2.1 Our contribution

In this article we have considered generalized version of the liar's dominating set problem, namely distance- d (m, ℓ) -liar's dominating set (distance- d (m, ℓ) -LDS). We prove that the distance- d (m, ℓ) -LDS problem is NP-complete by showing the following decision version of the distance- d (m, ℓ) -LDS problem is NP-complete.

Decision version of distance- d (m, ℓ) -LDS:

Input. A simple connected undirected graph $G = (V, E)$ with at least ℓ vertices and three positive integers $m, d(\leq |V| - 1)$, and $k(\leq |V|)$, where $m < \ell$.

Question. Does G have a distance- d (m, ℓ) -LDS of size at most k ?

We also prove that the distance- d (m, ℓ) -LDS problem cannot be approximated within a factor of $(\frac{1}{4} - \varepsilon) \ln |V|$ for any $\varepsilon > 0$, unless $NP \subseteq DTIME(|V|^{O(\log \log |V|)})$.

2.2 Organization

The remainder of the paper is organized as follows: In Section 3, we will prove that the decision version of distance-1 (m, ℓ) -LDS problem is NP-complete. Next, we prove the decision version of distance- d (m, ℓ) -LDS problem is NP-complete in Section 4. We have considered inapproximability results for distance-1 (m, ℓ) -LDS problem and distance- d (m, ℓ) -LDS problem in Section 5. More specifically, (i) in Subsection 5.1, we show that the distance-1 (m, ℓ) -LDS problem cannot be approximated within a factor of $(\frac{1}{2} - \varepsilon) \ln |V|$, for any $\varepsilon > 0$, unless $NP \subseteq DTIME(|V|^{O(\log \log |V|)})$, and (ii) in Subsection 5.2, we show that the distance- d (m, ℓ) -LDS problem cannot be approximated within a factor of $(\frac{1}{4} - \varepsilon) \ln |V|$, for any $d > 1$ and $\varepsilon > 0$, unless $NP \subseteq DTIME(|V|^{O(\log \log |V|)})$. Finally, we conclude the paper in Section 6.

3 Hardness of the distance-1 (m, ℓ) -LDS problem

In this section, we show that the distance-1 (m, ℓ) -LDS problem in graphs is NP-complete by reducing the *dominating set* (DS) problem to it, which is known to be NP-complete [2].

The decision versions of the distance-1 (m, ℓ) -LDS problem and DS problem are defined below.

Decision Version of the distance-1 (m, ℓ) -LDS problem:

Instance: An undirected graph $G = (V, E)$, m, ℓ , and a positive integer k .

Question: Does there exist a distance-1 (m, ℓ) -LDS L of G such that $|L| \leq k$?

Decision Version of the DS problem:

Instance: An undirected graph $G = (V, E)$ and a positive integer k .

Question: Does there exist a dominating set D of G such that $|D| \leq k$?

Theorem 1. *The decision version of the distance-1 (m, ℓ) -LDS problem is NP-complete.*

Proof. For any given set $L \subseteq V$ and a positive integer k , we can verify whether L is a distance-1 (m, ℓ) -LDS of size at most k or not in polynomial-time by checking both the conditions of distance-1 (m, ℓ) -LDS. Therefore, distance-1 (m, ℓ) -LDS is in NP.

Now, we prove the hardness of the distance-1 (m, ℓ) -LDS problem by reducing the decision version of DS problem, which is known to be NP-complete [2], to it. \square

Let $\langle G = (V, E), k \rangle$ be an instance of dominating set (DS) problem, where $G = (V, E)$ be an undirected graph and k is an integer. Also assume $V = \{v_1, v_2, \dots, v_n\}$. Now, we construct an instance $\langle G' = (V', E'), m, \ell \rangle$ to the decision version of distance-1 (m, ℓ) -LDS problem as follows:

$$V' = V^1 \cup V^2 \cup V^3$$

where

$$\begin{aligned} V^1 &= \{v_1^1, v_2^1, \dots, v_n^1\}, \\ V^2 &= \{v_1^2, v_2^2, \dots, v_{\ell-1}^2\}, \\ V^3 &= \{v_1^3, v_2^3\} \end{aligned}$$

and

$$E' = E^1 \cup E^2 \cup E^3 \cup E^4$$

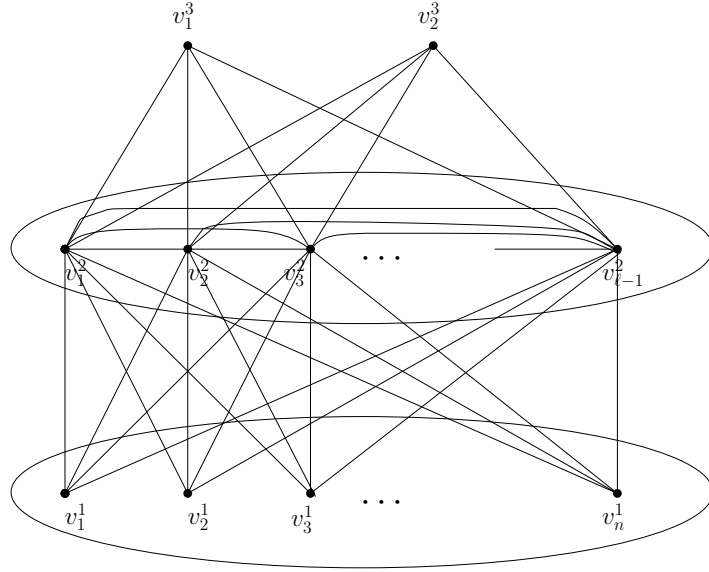


Figure 2: A graph $G' = (V', E')$ constructed from $G = (V, E)$.

where

$$\begin{aligned}
 E^1 &= \{(v_i^1, v_j^1) \mid (v_i, v_j) \in E\}, \\
 E^2 &= \{(v_i^2, v_j^2) \mid 1 \leq i < j \leq \ell - 1\}, \\
 E^3 &= \{(v_i^1, v_j^2) \mid 1 \leq i \leq n, 1 \leq j \leq \ell - 1\}, \\
 E^4 &= \{(v_i^2, v_1^3), (v_i^2, v_2^3) \mid 1 \leq i \leq \ell - 1\}
 \end{aligned}$$

Observe that, G' can be constructed in polynomial time and $|V'| = n + (\ell - 1) + 2 = n + \ell + 1$, where $n = |V|$ and $\ell < n$. The construction of G' from G is shown in Fig 2.

Claim 1: G has a dominating set of size at most k if and only if G' has a distance-1 (m, ℓ) -LDS of size at most $k + \ell$.

Proof: Let D be a dominating set of G and $|D| = k$. Let $L = \{v_i^1 \mid v_i \in D\} \cup V^2 \cup \{v_1^3\}$. Now, we will show that L is a distance-1 (m, ℓ) -LDS in G' .

(i) Observe that for each $v \in V'$, $|N_{G'}^1[v] \cap L| \geq m$ as $m < \ell$ and each $v \in V'$ is dominated by $\ell - 1$ vertices in V^2 .

(ii) Let u and v be any two distinct vertices V' .

Case 1: $\{u, v\} \in V^2 \cup V^3$, then $|(N_{G'}^1[u] \cup N_{G'}^1[v]) \cap L| = \{(V^2 \cup V^3 \setminus \{v_2^3\})\} \geq \ell$.

Case 2: $\{u, v\} \in V^1$, then $|(N_{G'}^1[u] \cup N_{G'}^1[v]) \cap L| \geq \ell$, because every vertex $v_i \in V^1$ is adjacent with $\ell - 1$ number of vertices in V^2 and D is a dominating set in G .

Case 3: $u \in V^1$ and $v \in V^2 \cup V^3$, then $|(N_{G'}^1[u] \cup N_{G'}^1[v]) \cap L| = |\{c\} \cup V^2| \geq \ell$, where $c \in \{v_i^1 \mid v_i \in D\} \cap N_{G'}^1[v]$.

Thus L is a distance-1 (m, ℓ) -LDS in G' and $|L| \leq k + \ell$.

Conversely, let L is a distance-1 (m, ℓ) -LDS of G' of size at most $k + \ell$. Since $|(N_{G'}^1[v_1^3] \cup N_{G'}^1[v_2^3]) \cap L| \geq \ell$, there must be ℓ vertices from $V^2 \cup V^3$ in L (see Fig. 2). Let $D' = L \setminus (V^2 \cup V^3)$ and $D = \{v_i \in V \mid v_i^1 \in V' \cap D'\}$. It remains to prove that D is a dominating set of the graph G . Suppose that D is not a dominating set of G . Then there exist only one vertex $v \in V$ such that $D \cap N_G^1[v] = \phi$. If more than one vertex exists say, $\{u, v\} \in V$ such that $D \cap N_G^1[u] = \phi$ and $D \cap N_G^1[v] = \phi$, then $|(N_{G'}^1[u] \cup N_{G'}^1[v]) \cap L| \leq |\{v_1^2, v_2^2, \dots, v_{\ell-1}^2\}| < \ell$, which is a contradiction to the fact that L is a distance-1 (m, ℓ) -LDS of G' . Therefore, if D is not a dominating set of G then there can be at most one vertex $v \in V$ such that $D \cap N_G^1[v] = \phi$. Observe that $V^2 \cup V^3 \subseteq L$. On contrary assume that $V^2 \cup V^3 \not\subseteq L$ i.e., $|(V^2 \cup V^3) \cap L| \leq \ell$. Then there exist a vertex $u \in V^2 \cup V^3$ and $u \notin L$.

Case 1: $u \in V^3$ and $u \notin L$. Now, $|(N_{G'}^1[v] \cup N_{G'}^1[u]) \cap L| = |\{v_1^2, v_2^2, \dots, v_{\ell-1}^2\}| = \ell - 1$, which is a contradiction to the fact that L is a distance-1 (m, ℓ) -LDS of graph G' .

Case 2: $u \in V^2$ and $u \notin L$. Now, $|(N_{G'}^1[v] \cup N_{G'}^1[v_1^3]) \cap L| \leq (|V^2| - 1) + |\{v_1^3\}| = \ell - 1$, which is a contradiction to the fact that L is a distance-1 (m, ℓ) -LDS of graph G' .

Thus, $V^2 \cup V^3 \subseteq L$. Delete v_2^3 from L and introduce v , i.e., $L = (L \setminus \{v_2^3\}) \cup \{v\}$. So D is a dominating set of G and $|D| \leq k$.

Therefore, we conclude, distance-1 (m, ℓ) -LDS problem is NP-complete. \square

4 Hardness of the distance- d (m, ℓ) -LDS problem

In this section, we show that distance- d (m, ℓ) -LDS problem is NP-complete. Here, we prove that the decision version of the distance- d (m, ℓ) -LDS problem is NP-complete, which leads to the claim of this section. For fixed constant $d \geq 2$, the decision version of the distance- d (m, ℓ) -LDS problem is defined as follows.

Instance: An undirected connected graph $G = (V, E)$ with $|V| \geq \ell$ and two positive integers $k \leq |V|$, and $d \leq |V| - 1$.

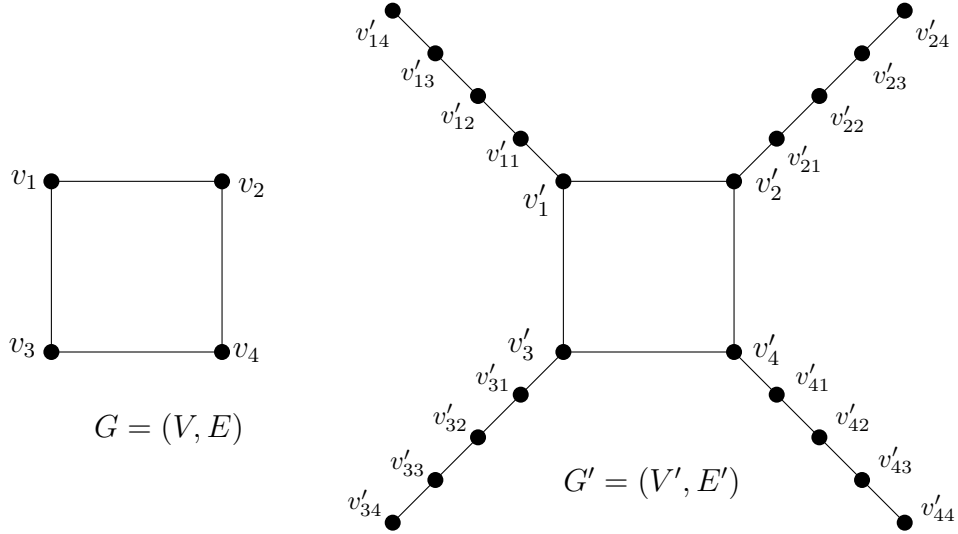


Figure 3: Illustration of polynomial time reduction from an instance of the decision version of distance-1 (m, ℓ) -LDS problem to an instance of the decision version of distance- d (m, ℓ) -LDS problem

Question: Does G have a distance- d (m, ℓ) -LDS of size at most k ?

We prove that decision version of distance- d (m, ℓ) -LDS problem ($d \geq 2$) is NP-complete by reducing the decision version of the distance-1 (m, ℓ) -LDS problem to it in polynomial time. Note that distance-1 (m, ℓ) -LDS problem is NP-complete (see Section 3). Recall, the decision version of distance-1 (m, ℓ) -LDS problem:

Instance: An undirected connected graph $G = (V, E)$ with $|V| \geq \ell$ and a positive integer $k \leq |V|$.

Question: Does G has a distance-1 (m, ℓ) -LDS of size at most k ?

Theorem 2. *The decision version of the distance- d (m, ℓ) -LDS problem is NP-complete.*

Proof. The decision version of distance- d (m, ℓ) -LDS problem is in NP as for a given certificate (a subset of V) we can verify whether it is satisfying both the conditions of distance- d (m, ℓ) -LDS or not in polynomial time.

We now describe a polynomial time reduction from an arbitrary instance of decision version of distance-1 (m, ℓ) -LDS to an instance of decision version of distance- d (m, ℓ) -LDS.

Let $G = (V = \{v_1, v_2, \dots, v_n\}, E)$ be an arbitrary instance of the decision version of distance-1 (m, ℓ) -LDS problem. We construct an instance, a graph

$G' = (V', E')$, of the decision version of distance- d (m, ℓ) -LDS problem as follows:

$$V' = \{v'_i \mid v_i \in V\} \cup \bigcup_{v_i \in V} \{v'_{i1}, v'_{i2}, \dots, v'_{id-1}\} \text{ (see Fig.3 for an example)}$$

$$E' = \{(v'_i, v'_j) \mid (v_i, v_j) \in E\} \cup \bigcup_{v_i \in V} \{(v'_i, v'_{i1}), (v'_{i1}, v'_{i2}), \dots, (v'_{id-2}, v'_{id-1})\}$$

Claim 2: G has a distance-1 (m, ℓ) -LDS of cardinality at most k if and only if G' has a distance- d (m, ℓ) -LDS of cardinality at most k .

Necessity: Let L be a distance-1 (m, ℓ) -LDS of G such that $|L| \leq k$. Let $L' = \{v'_i \in V' \mid v_i \in L\}$. We can argue that L' is a distance- d (m, ℓ) -LDS in G' and $|L'| \leq k$. Since $|L'| = |L|$ and $|L| \leq k$, so $|L'| \leq k$. As each vertex $v \in V$ satisfies distance-1 (m, ℓ) -LDS properties and each vertex in G' is at most $d-1$ distance away from a vertex in L' , L' suffices to ensure distance- d (m, ℓ) -LDS in graph G' for $d \geq 1$.

Sufficiency: Let L' be a distance- d (m, ℓ) -LDS in G' such that $|L'| \leq k$. We shall show that, by updating (i.e., removing or replacing) some of the vertices in L' , at most k vertices from $\{v'_1, v'_2, \dots, v'_n\}$ can be chosen such that the set of corresponding vertices in V is a distance-1 (m, ℓ) -LDS in G . Let $L'' = L'$. For each vertex $v'_{ij} \in V'$, ($1 \leq j \leq d-1$ and $1 \leq i \leq n$) we do the following: if $v'_{ij} \in L''$, then replace it with its associated vertex v'_i if v'_i is not already in L'' , otherwise, replace it with any vertex in $N_{G'}^1[v'_i] \cap \{v'_1, v'_2, \dots, v'_n\}$ which is not in L'' . If all the vertices of $N_{G'}^1[v'_i] \cap \{v'_1, v'_2, \dots, v'_n\}$ are in L'' (i.e., $(N_{G'}^1[v'_i] \cap \{v'_1, v'_2, \dots, v'_n\}) \subseteq L''$), then remove v'_{ij} from L'' . Therefore, $|L''| \leq k$. Let $L = \{v_i \in V \mid v'_i \in L''\}$. Now, we prove that L is a distance-1 (m, ℓ) -LDS in G such that $|L| \leq k$.

Since $|L''| \leq k$, then $|L| \leq k$. We first prove the first condition (i.e., for every $v \in V$, $|N_G^1[v] \cap L| \geq m$) of distance-1 (m, ℓ) -LDS.

Consider a vertex $v'_i \in V'$, for some $1 \leq i \leq n$, let s be the number of vertices in L' from the set $\{v'_{i1}, v'_{i2}, \dots, v'_{id-1}\}$.

Case 1. $s = 0$. Since L' is distance- d (m, ℓ) -LDS, there must exist at least m vertices, say $\{v''_1, v''_2, \dots, v''_m\}$ in $\{v'_1, v'_2, \dots, v'_n\} \cap L'$ such that $\{v''_1, v''_2, \dots, v''_m\} \subseteq N_{G'}^d[v'_{i,d-1}]$, otherwise, L' is not a feasible solution as $v'_{i,d-1}$ does not have m distance- d (m, ℓ) -dominators. Therefore, $|N_{G'}^1[v'_i] \cap (\{v'_1, v'_2, \dots, v'_n\} \cap L'')| \geq m$.

Case 2. $s \geq 1$. Let $v'_{ij_1}, v'_{ij_2}, \dots, v'_{ij_t} \in L'$, for some $1 \leq j_1, j_2, \dots, j_t \leq d-1$. By our construction of L'' each vertex in $\{v'_{ij_1}, v'_{ij_2}, \dots, v'_{ij_t}\}$ is replaced by one of the vertices in $N_{G'}^1[v'_i] \cap \{v'_1, v'_2, \dots, v'_n\}$. Therefore, in this case also $|N_{G'}^1[v'_i] \cap (\{v'_1, v'_2, \dots, v'_n\} \cap L'')| \geq m$.

Thus, by our construction of L from L'' , $|N_G^1[v_i] \cap L| \geq m$ is true.

Now we prove the second condition of distance-1 (m, ℓ) -LDS (i.e., for every pair of distinct vertices $u, v \in V$, $|(N_G^1[u] \cup N_G^1[v]) \cap L| \geq \ell$).

Let v_i and v_j be two distinct vertices in G . Consider the vertices v'_{id-1} and v'_{jd-1} in G' . As L' is a distance- d (m, ℓ) -LDS of G' , it satisfies the second property of distance- d (m, ℓ) -LDS in G' . Thus there exist at least ℓ dominators

dominating v'_{id-1} and v'_{jd-1} in L' , i.e., $|(N_{G'}^d[v'_{id-1}] \cup N_{G'}^d[v'_{jd-1}]) \cap L'| \geq \ell$. These dominators are either from $N_G^1[v_i] \cup N_G^1[v_j]$ or from $\{v'_{i1}, v'_{i2}, \dots, v'_{id-1}\}$ and/or from $\{v'_{j1}, v'_{j2}, \dots, v'_{jd-1}\}$. As per our construction of L'' from L' , we are replacing each dominator in $\{v'_{i1}, v'_{i2}, \dots, v'_{id-1}\} \cup \{v'_{j1}, v'_{j2}, \dots, v'_{jd-1}\}$ (if any) by a vertex in $(N_{G'}^1[v_i] \cup N_{G'}^1[v_j]) \cap \{v'_1, v'_2, \dots, v'_n\}$.

Since G is connected and $|V| \geq \ell$, so is G' . Therefore, L'' contains at least ℓ vertices from $(N_{G'}^1[v_i] \cup N_{G'}^1[v_j]) \cap \{v'_1, v'_2, \dots, v'_n\}$, i.e., $|(N_{G'}^1[v_i] \cup N_{G'}^1[v_j]) \cap \{v'_1, v'_2, \dots, v'_n\} \cap L''| \geq \ell$. Therefore, according to the construction of L from L'' , $|(N_G^1[v_i] \cup N_G^1[v_j]) \cap L| \geq \ell$. Thus, L is a distance-1 (m, ℓ) -LDS of the graph G having cardinality at most k .

Therefore, the decision version of distance- d (m, ℓ) -LDS problem is NP-complete. \square

5 Inapproximability results

5.1 Inapproximability of distance-1 (m, ℓ) -LDS

In this section, we shall prove that the distance-1 (m, ℓ) -LDS problem cannot be approximated within a factor of $(\frac{1}{2} - \varepsilon) \ln(|V|)$ for any $\varepsilon > 0$, unless $NP \subseteq DTIME(|V|^{\log \log |V|})$. We argue the claim by showing that if distance-1 (m, ℓ) -LDS can be approximated within a factor of $(\frac{1}{2} - \varepsilon) \ln(|V|)$ for any $\varepsilon > 0$ in a graph G' , then the dominating set problem can be approximated within a factor of $(1 - \varepsilon) \ln(|V|)$ for any $\varepsilon > 0$.

Theorem 3. *[2] Minimum dominating set problem cannot be approximated within a factor of $(1 - \varepsilon) \ln(|V|)$ for any $\varepsilon > 0$, unless $NP \subseteq DTIME(|V|^{\log \log |V|})$.*

Theorem 4. *Minimum distance-1 (m, ℓ) -LDS problem cannot be approximated within a factor of $(\frac{1}{2} - \varepsilon) \ln(|V|)$ for any $\varepsilon > 0$, unless $NP \subseteq DTIME(|V|^{\log \log |V|})$.*

Proof. Let G be a simple graph. Consider the construction of the graph G' for any given graph G as discussed in Section 3. As per our construction, we proved that dominating set problem can be polynomially reducible to distance-1 (m, ℓ) -LDS problem.

Let D^* and L^* be the optimal DS and distance-1 (m, ℓ) -LDS in G and G' , with cardinalities $\gamma_{ds}(G)$ and $\gamma_{LR}(G')$, respectively. Now we can argue the following claim: $\gamma_{LR}(G') = \gamma_{ds}(G) + \ell$. The inequality $\gamma_{LR}(G') \leq \gamma_{ds}(G) + \ell$ is trivial as per our construction in Section 3. On the other hand, $\gamma_{LR}(G') \geq \gamma_{ds}(G) + \ell$ follows from the sufficiency proof of Claim 1 in Section 3. So given a dominating set D of G , one can find a distance-1 (m, ℓ) -LDS L of G' such that $|L| = |D| + \ell$. Now, $\frac{|L|}{|L^*|} = \frac{|D| + \ell}{|D^*| + \ell} \geq \frac{1}{2} \frac{|D|}{|D^*|}$.

Suppose there exists a polynomial time algorithm that approximates distance-1 (m, ℓ) -LDS problem within a factor of $(\frac{1}{2} - \varepsilon) \ln N$ for graphs with N vertices. As per our construction of the graph G' from G (see Fig. 2), G' contains, $N = n + \ell + 1 \leq 2n$ for $n \geq 2$ vertices, where n is the total number of vertices in G and $\ell < n$. Therefore,

$$\begin{aligned}
\frac{1}{2} \frac{|D|}{|D^*|} &\leq \left(\frac{1}{2} - \varepsilon\right) \ln N \\
\implies \frac{|D|}{|D^*|} &\leq (1 - 2\varepsilon) \ln 2n \\
&= (1 - 2\varepsilon) \ln n \left(1 + \frac{\ln 2}{\ln n}\right)
\end{aligned}$$

For sufficiently large n , the term $(1 + \frac{\ln 2}{\ln n})$ can be bounded by $1 + \frac{\varepsilon}{5}$, where $\varepsilon \geq \frac{5 \ln 2}{\ln n}$. Now we have

$$(1 - 2\varepsilon) \ln n \left(1 + \frac{\ln 2}{\ln n}\right) \leq (1 - \varepsilon') \ln n,$$

where $\varepsilon' < \frac{9}{5}\varepsilon + \frac{2}{5}\varepsilon^2$. Therefore, for an arbitrary graph, we can approximate the dominating set problem by a factor of $(1 - \varepsilon') \ln n$, which leads to a contradiction to Theorem 3. Thus, the minimum distance-1 (m, ℓ) -LDS problem cannot be approximated within a factor of $(\frac{1}{2} - \varepsilon) \ln(|V|)$ for any $\varepsilon > 0$, unless $NP \subseteq DTIME(|V|^{\log \log |V|})$. \square

5.2 Inapproximability of the distance- d (m, ℓ) -LDS problem

In this section, we give a lower bound on the approximation ratio of any approximation algorithm for the distance- d (m, ℓ) -LDS problem by providing an approximation preserving reduction from the distance-1 (m, ℓ) -LDS problem.

Theorem 5. *For a given undirected graph $G = (V, E)$, the distance- d (m, ℓ) -LDS problem cannot be approximated within a factor of $(\frac{1}{4} - \varepsilon) \ln |V|$, for any fixed constant $d \geq 2$ and $\varepsilon > 0$, unless $NP \subseteq DTIME(|V|^{O(\log \log |V|)})$.*

Proof. Let $G = (V, E)$ be an arbitrary instance of the distance-1 (m, ℓ) -LDS problem with n vertices. Given $G = (V, E)$, we construct a graph $G' = (V', E')$, an instance of the distance- d (m, ℓ) -LDS problem as described in Section 4. Let L^* and L_d^* be the optimal distance-1 (m, ℓ) -LDS and distance- d (m, ℓ) -LDS in G and G' , with cardinalities $\gamma_{LR}(G)$ and $\gamma_{LR}^d(G')$, respectively. Now we can argue the following claim: $\gamma_{LR}^d(G') = \gamma_{LR}(G)$. The inequality $\gamma_{LR}^d(G') \leq \gamma_{LR}(G)$ is trivial as every distance-1 (m, ℓ) -LDS of G is a distance- d (m, ℓ) -LDS in G' . On the other hand, $\gamma_{LR}^d(G') = |L_d^*| \geq |L|$ follows from the sufficiency proof of Claim 2 in Section 4.

Given any distance-1 (m, ℓ) -LDS L of G , one can find a distance- d (m, ℓ) -LDS L_d of G' with $|L_d| = |L|$. Suppose there exist a polynomial time algorithm to approximate distance- d (m, ℓ) -LDS problem within a factor of $(\frac{1}{4} - \varepsilon) \ln |V'|$, where $|V'| = n + n(d-1) \leq n^2$ (see Section 4). Now $\frac{|L|}{|L^*|} = \frac{|L_d|}{|L_d^*|} \leq (\frac{1}{4} - \varepsilon) \ln n^2 = (\frac{1}{2} - 2\varepsilon) \ln n \leq (\frac{1}{2} - \varepsilon') \ln n$, where $\varepsilon' \leq 2\varepsilon$. Therefore, the result follows from Theorem 4. \square

6 Conclusion

In this paper, we showed that the distance-1 (m, ℓ) -liar's dominating set (distance-1 (m, ℓ) -LDS) problem is NP-complete and proved that it cannot be approximated within a factor of $(\frac{1}{2} - \varepsilon) \ln |V|$, unless $NP \subseteq DTIME(|V|^{O(\log \log |V|)})$, where V is the vertex set of input graph. We also proved that distance- d (m, ℓ) -liar's dominating set (distance- d (m, ℓ) -LDS) problem is NP-hard and proved that the problem cannot be approximated within a factor of $(\frac{1}{4} - \varepsilon) \ln |V|$, unless $NP \subseteq DTIME(|V|^{O(\log \log |V|)})$, where V is the vertex set of input graph. Both the problems are generalized version of the liar's dominating set problem available in literature.

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